

# Center of $U(\mathfrak{n})$ , Cascade of Orthogonal Roots and a Construction of Lipsman–Wolf

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*Dedicated to Joe, a special friend and valued colleague*

**Abstract.** Let  $G$  be a complex simply-connected semisimple Lie group and let  $\mathfrak{g} = \text{Lie } G$ . Let  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$  be a triangular decomposition of  $\mathfrak{g}$ . One readily has that  $\text{Cent } U(\mathfrak{n})$  is isomorphic to the ring  $S(\mathfrak{n})^{\mathfrak{n}}$  of symmetric invariants. Using the cascade  $\mathcal{B}$  of strongly orthogonal roots, some time ago we proved (see [K]) that  $S(\mathfrak{n})^{\mathfrak{n}}$  is a polynomial ring  $\mathbb{C}[\xi_1, \dots, \xi_m]$  where  $m$  is the cardinality of  $\mathcal{B}$ . The authors in [LW] introduce a very nice representation-theoretic method for the construction of certain elements in  $S(\mathfrak{n})^{\mathfrak{n}}$ . A key lemma in [LW] is incorrect but the idea is in fact valid. In our paper here we modify the construction so as to yield these elements in  $S(\mathfrak{n})^{\mathfrak{n}}$  and use the [LW] result to prove a theorem of Tony Joseph.

**Key words:** cascade of orthogonal roots, Borel subgroups, nilpotent coadjoint action.

**MSC (2010) codes:** representation theory, invariant theory.

## 1. Introduction

**1.1.** Let  $\mathfrak{g}$  be a complex semisimple Lie algebra and let

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$$

be a fixed triangular decomposition of  $\mathfrak{g}$ . Let  $\Delta \subset \mathfrak{h}^*$  be the set of  $\mathfrak{h}$  roots in  $\mathfrak{g}$ . The Killing form  $(x, y)$  on  $\mathfrak{g}$ , denoted by  $\mathcal{K}$ , induces a nonsingular bilinear form  $(\mu, \nu)$  on  $\mathfrak{h}^*$ . For each  $\varphi \in \Delta$  let  $e_\varphi \in \mathfrak{g}$  be a corresponding root vector. The root vectors can and will be chosen so that  $(e_\varphi, e_{-\varphi}) = 1$  for all roots  $\varphi$ .

If  $\mathfrak{s} \subset \mathfrak{g}$  is any subspace stable under  $\text{ad } \mathfrak{h}$  let

$$\Delta(\mathfrak{s}) = \{\varphi \in \Delta \mid e_\varphi \in \mathfrak{s}\}.$$

The set  $\Delta_+$  of positive roots is then chosen so that  $\Delta_+ = \Delta(\mathfrak{n})$ , and one puts  $\Delta_- = -\Delta_+$ . If  $\mathfrak{s}$  is a Lie subalgebra, then  $S(\mathfrak{s})$  and  $U(\mathfrak{s})$  are respectively the symmetric and enveloping algebras of  $\mathfrak{s}$ . Our concern here is with the case where  $\mathfrak{s} = \mathfrak{n}$ .

Let  $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$  so that  $\mathfrak{b}$  is a Borel subalgebra of  $\mathfrak{g}$ . Let  $G$  be a Lie group such that  $\text{Lie } G = \mathfrak{g}$  and let  $H, N, B$  be Lie subgroups corresponding, respectively, to  $\mathfrak{h}, \mathfrak{n}, \mathfrak{b}$ . Then  $S(\mathfrak{n})$  is a  $B$ -module since  $B = HN$  normalizes  $N$ . Let  $m$  be the maximal number of strongly orthogonal roots. Then we proved the following some time ago, generalizing a result of Dixmier (case where  $\mathfrak{g}$  is of type  $A_\ell$ ),

**Theorem A.** *There exists  $\xi_i \in S(\mathfrak{n})^N, i = 1, \dots, m$ , so that*

$$S(\mathfrak{n})^N = \mathbb{C}[\xi_1, \dots, \xi_m]$$

*is a polynomial ring in  $m$ -generators. Furthermore,*

$$S(\mathfrak{n})^N \cong \text{Cent } U(\mathfrak{n})$$

*so that one has a similar statement for  $\text{Cent } U(\mathfrak{n})$ .*

We will present an algebraic-geometric proof of a much stronger statement than Theorem A and relate it to a representation-theoretic construction, due to Lipsman–Wolf, of certain elements in  $S(\mathfrak{n})^N$ . See [K], [LW]. A key tool is the cascade  $\mathcal{B} = \{\beta_1, \dots, \beta_m\}$  of orthogonal roots which will now be defined.

**1.2.** Let  $\Pi \subset \Delta_+$  be the set of simple positive roots. For any  $\varphi \in \Delta_+$  and  $\alpha \in \Pi$  there exists a nonnegative integer  $n_\alpha(\varphi)$  such that

$$\varphi = \sum_{\alpha \in \Pi} n_\alpha(\varphi) \alpha.$$

Let

$$\Pi(\varphi) = \{\alpha \in \Pi \mid n_\alpha(\varphi) > 0\}.$$

Then  $\Pi(\varphi)$  is a connected subset of  $\Pi$  and hence defines a simple Lie subalgebra  $\mathfrak{g}(\varphi)$  of  $\mathfrak{g}$ . We will say that  $\varphi$  is locally high if  $\varphi$  is the highest root of  $\mathfrak{g}(\varphi)$ . Obviously the highest roots of all the simple components of  $\mathfrak{g}$  are locally high.

**Remark 1.** If  $\mathfrak{g}$  is of type  $A_\ell$ , but only in this case, are all  $\varphi \in \Delta_+$  locally high.

Let  $\varphi \in \Delta_+$  be locally high and let

$$\Pi(\varphi)^o = \{\alpha \in \Pi(\varphi) \mid (\alpha, \varphi) = 0\};$$

let  $\mathfrak{g}(\varphi)^o$  be the semisimple Lie algebra having  $\Pi(\varphi)^o$  as its set of simple roots. We will then say that a root  $\varphi' \in \Delta_+$  is an offspring of  $\varphi$  if  $\varphi'$  is the highest root of a simple component of  $\mathfrak{g}(\varphi)^o$ .

**Remark 2.** One notes that an offspring of a locally high root  $\varphi$  is again locally high and that it is strongly orthogonal to  $\varphi$ .

A sequence of positive roots

$$C = \{\beta'_1, \dots, \beta'_k\}$$

will be called a cascade chain if  $\beta'_1$  is a highest root of a simple component of  $\mathfrak{g}$ , and if  $1 < j \leq k$ , then  $\beta'_j$  is an offspring of  $\beta'_{j-1}$ . Now let  $\mathcal{B}$  be the set of all positive roots  $\beta$  which are members of some cascade chain. Let  $W$  be the Weyl of  $(\mathfrak{h}, \mathfrak{g})$ .

**Theorem 1.** *The cardinality of  $\mathcal{B}$  is  $m$  and*

$$\mathcal{B} = \{\beta_1, \dots, \beta_m\}$$

*is a maximal set of strongly orthogonal roots. Furthermore, if  $s_{\beta_i}$  is the  $W$ -reflection of  $\mathfrak{h}$  corresponding to  $\beta_i$ , then the long element  $w_o$  of  $W$  may be given by*

$$w_o = s_{\beta_1} \cdots s_{\beta_m}. \quad (1.1)$$

$\mathcal{B}$  is the cascade of orthogonal roots.

**1.3.** One has the vector space direct sum

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}. \quad (1.2.)$$

Let  $P : \mathfrak{g} \rightarrow \mathfrak{n}$  be the projection defined by (1.2). Since  $\mathfrak{b}$  is the  $\mathcal{K}$ -orthogonal subspace to  $\mathfrak{n}$  in  $\mathfrak{g}$  we may identify  $\mathfrak{n}_-$  with the dual space  $\mathfrak{n}^*$  to  $\mathfrak{n}$ , so that for  $v \in \mathfrak{n}_-$  and  $x \in \mathfrak{n}$ , one has  $\langle v, x \rangle = (v, x)$ . The coadjoint action of  $N$  on  $\mathfrak{n}_-$  may then be given so that if  $u \in N$ , then on  $\mathfrak{n}_-$

$$\text{Coad } u = P \text{Ad } u. \quad (1.3.)$$

In fact, using (1.2) the coadjoint action of  $N$  on  $\mathfrak{n}_-$  extends to an action of  $B$  on  $\mathfrak{n}_-$ , so that if  $b \in B$  and  $v \in \mathfrak{n}_-$ , one has  $b \cdot v = P \text{Ad } b(v)$ . In addition we can regard  $S(\mathfrak{n})$  as the ring of polynomial functions on  $\mathfrak{n}_-$ . Since  $B$  normalizes  $N$  the natural action of  $N$  on  $S(\mathfrak{n})$  extends to an action of  $B$  on  $S(\mathfrak{n})$  where if  $f \in S(\mathfrak{n})$ ,  $b \in B$ , and  $v \in \mathfrak{n}_-$ , one has

$$(b \cdot f)(v) = f(b^{-1} \cdot v). \quad (1.4)$$

Recalling  $m = \text{card } \mathcal{B}$ , let  $\mathfrak{r}$  be the commutative  $m$ -dimensional subalgebra of  $\mathfrak{n}$  spanned by  $e_\beta$  for  $\beta \in \mathcal{B}$  and let  $R \subset N$  be the commutative unipotent subgroup corresponding to  $\mathfrak{r}$ . In the dual space let  $\mathfrak{r}_- \subset \mathfrak{n}_-$  be the span of  $e_{-\beta}$  for  $\beta \in \mathcal{B}$ . For any  $z \in \mathfrak{r}_-$ ,  $\beta \in \mathcal{B}$ , let  $a_\beta(z) \in \mathbb{C}$  be defined so that

$$z = \sum_{\beta \in \mathcal{B}} a_\beta(z) e_{-\beta}, \quad (1.5)$$

and let

$$\mathfrak{r}_-^\times = \{\tau \in \mathfrak{r}_- \mid a_\beta(\tau) \neq 0, \forall \beta \in \mathcal{B}\}.$$

As an algebraic subvariety of  $\mathfrak{n}_-$  clearly

$$\mathfrak{r}_-^\times \cong (\mathbb{C}^\times)^m. \quad (1.6)$$

Also for any  $z \in \mathfrak{n}_-$  let  $O_z$  be the  $N$ -coadjoint orbit containing  $z$ . Let  $N_z \subset N$  be the coadjoint isotropy subgroup at  $z$  and let  $\mathfrak{n}_z = \text{Lie } N_z$ . Since the action is algebraic,  $N_z$  is connected and hence as  $N$ -spaces

$$O_z \cong N/N_z. \quad (1.7)$$

**Theorem 2.** *Let  $\tau \in \mathfrak{r}_-^\times$ . Then (independent of  $\tau$ )  $N_\tau = R$  so that (1.7) becomes*

$$O_\tau \cong N/R. \quad (1.8)$$

*In particular*

$$\dim O_\tau = \dim \mathfrak{n} - m \quad (1.9)$$

*and  $O_\tau$  is a maximal dimensional coadjoint orbit of  $N$ .*

Now consider the action of  $B$  on  $\mathfrak{n}_-$ . In particular consider the action of  $H$  on  $\mathfrak{n}_-$ . Obviously

$$\mathfrak{r}_-^\times \cong (\mathbb{C}^\times)^m, \quad (1.10)$$

and furthermore  $\mathfrak{r}_-^\times$  is an orbit of  $H$ . In addition  $H$  permutes the maximal  $N$ -coadjoint orbits  $O_\tau$ ,  $\tau \in \mathfrak{r}_-^\times$ . More precisely,

**Theorem 3.** *For any  $a \in H$  and  $\tau \in \mathfrak{r}_-^\times$ , one has*

$$a \cdot O_\tau = O_{a \cdot \tau}. \quad (1.11)$$

**1.4.** If  $V$  is an affine variety,  $A(V)$  will denote its corresponding affine ring of functions. Note that  $S(\mathfrak{n}) = A(\mathfrak{n}_-)$ . Let  $Q(\mathfrak{n}_-)$  be the quotient field of  $S(\mathfrak{n})$ .

**Theorem 4.** *There exists a unique Zariski open nonempty orbit  $X$  of  $B$  on  $\mathfrak{n}_-$ . In particular*

$$\overline{X} = \mathfrak{n}_-. \quad (1.12)$$

*Furthermore  $X$  is an affine variety so that*

$$S(\mathfrak{n}_-) \subset A(X) \subset Q(\mathfrak{n}_-). \quad (1.13)$$

*Moreover  $\mathfrak{n}_-^\times \subset X$ , and in fact one has a disjoint union*

$$X = \sqcup_{\tau \in \mathfrak{r}_-^\times} O_\tau \quad (1.14)$$

so that all  $N$ -coadjoint orbits in  $X$  are maximal and isomorphic to  $N/R$ .

Let  $\Lambda \subset \mathfrak{h}^*$  be the  $H$ -weight lattice and let  $\Lambda_{\text{ad}} \subset \Lambda$  be the root lattice. Let  $\Lambda_{\mathcal{B}} \subset \Lambda_{\text{ad}}$  be the sublattice generated by the cascade  $\mathcal{B}$ . Since the elements of  $\mathcal{B}$  are mutually orthogonal note that

$$\Lambda_{\mathcal{B}} = \bigoplus_{\beta \in \mathcal{B}} \mathbb{Z} \beta \quad (1.15)$$

is a free  $\mathbb{Z}$ -module of rank  $m$ .

If  $M$  is an  $H$ -module, let  $\Lambda(M) \subset \Lambda$  be the set of  $H$ -weights occurring in  $M$ . Note that if  $M$  is a  $B$ -module, then  $M^N$  is still an  $H$ -module. Recalling the definition of  $\mathfrak{r}_-^\times$  and (1.6), note that

$$\Lambda(A(\mathfrak{r}_-^\times)) = \Lambda_{\mathcal{B}} \quad (1.16)$$

and each weight occurs with multiplicity 1.

We can now give more information about  $X$  and its affine ring  $A(X)$ . Define a  $B$  action on  $\mathfrak{r}_-^\times$  by extending the  $H$ -action so that  $N$  operates trivially. Next define a  $B$ -action on  $N/R$ , extending the  $N$ -action by letting  $H$  operate by conjugation, noting that  $H$  normalizes both  $N$  and  $R$ . With these structures and the original action on  $X$ , we have the following.

**Theorem 5.** *One has a  $B$ -isomorphism*

$$X \rightarrow N/R \times \mathfrak{r}_-^\times$$

*of affine varieties so that as  $B$ -modules*

$$A(X) \cong A(N/R) \otimes A(\mathfrak{r}_-^\times). \quad (1.17)$$

*Furthermore, taking  $N$ -invariants, one has an  $H$ -module isomorphism*

$$A(X)^N \cong A(\mathfrak{r}_-^\times) \quad (1.18)$$

*so that, by (1.16),*

$$\Lambda(A(X)^N) = \Lambda_{\mathcal{B}} \quad (1.19)$$

*and each  $H$ -weight occurs with multiplicity 1.*

Recalling (1.13) one has the  $N$ -invariant inclusions

$$S(\mathfrak{n})^N \subset A(X)^N \subset Q(n_-)^N \quad (1.20)$$

of  $H$ -modules so that

$$\Lambda(S(\mathfrak{n})^N) \subset \Lambda(A(X)^N) \subset \Lambda(Q(n_-)^N). \quad (1.21)$$

But since  $S(\mathfrak{n})$  is a unique factorization domain, any  $u \in Q(\mathfrak{n}_-)$  may be uniquely written, up to scalar multiplication as

$$u = f/g \quad (1.22)$$

where  $f$  and  $g$  are prime to one another. Furthermore, it is then immediate (since  $N$  is unipotent) that if  $u$  is  $N$ -invariant, one has  $f, g \in S(\mathfrak{n})^N$ . If, in addition,  $u$  is an  $H$ -weight vector, the same is true of  $f$  and  $g$  so that, using Theorem 5, one readily concludes the following.

**Theorem 6.** *Every  $H$ -weight in  $\Lambda(S(\mathfrak{n})^N)$  occurs with multiplicity 1 in  $S(\mathfrak{n})^N$ . In fact  $\Lambda(Q(\mathfrak{n}_-)) = \Lambda_{\mathcal{B}}$  and every weight  $\gamma$  in  $\Lambda(Q(\mathfrak{n}_-))$  occurs with multiplicity 1 in  $Q(\mathfrak{n}_-)^N$  and is of the form*

$$\gamma = \nu - \mu \quad (1.23)$$

where  $\mu, \nu \in \Lambda(S(\mathfrak{n})^N)$ .

For any  $\gamma \in \Lambda_{\mathcal{B}}$  let  $\xi_{\gamma} \in Q(\mathfrak{n}_-)^N$  be the unique (up to scalar multiplication)  $H$ -weight vector with weight  $\gamma$ . Thus if  $\gamma \in \Lambda_{\mathcal{B}}$ , we may uniquely write (up to scalar multiplication

$$\xi_{\gamma} = \xi_{\nu} / \xi_{\mu} \quad (1.24)$$

where  $\mu, \nu \in \Lambda(S(\mathfrak{n})^N)$  and  $\xi_{\nu}$  and  $\xi_{\mu}$  are prime to one another. Let

$$\Lambda_{\text{dom}} = \{\lambda \in \Lambda \mid \lambda \text{ be a dominant weight}\}.$$

**Remark 3.** By the multiplicity 1-condition note that if  $\nu \in \Lambda(S(\mathfrak{n})^N)$ , then  $\xi_{\nu}$  is necessarily a homogeneous polynomial. Define  $\deg \nu$  so that  $\xi_{\nu} \in S^{\deg \nu}(\mathfrak{n})$ . Furthermore, clearly  $\xi_{\nu}$  is then a highest weight vector of an irreducible  $\mathfrak{g}$ -module in  $S^{\deg \nu}(\mathfrak{g})$  and in particular  $\nu \in \Lambda_{\text{dom}}$ . That is,

$$\Lambda(S(\mathfrak{n})^N) \subset \Lambda_{\text{dom}} \cap \Lambda_{\mathcal{B}}. \quad (1.25)$$

**1.5.** If  $\nu \in \Lambda(S(\mathfrak{n})^N)$ , it follows easily from the multiplicity-1 condition and the uniqueness of prime factorization that all the prime factors of  $\xi_{\nu}$  are again weight vectors in  $S(\mathfrak{n})^N$ . Let

$$\mathcal{P} = \{\nu \in \Lambda(S(\mathfrak{n})^N) \mid \xi_{\nu} \text{ be a prime polynomial in } S(\mathfrak{n})^N\}. \quad (1.26)$$

We can then readily prove

**Theorem 7.** *One has  $\text{card } \mathcal{P} = m$  where, we recall  $m = \text{card } \mathcal{B}$ , so that we can write*

$$\mathcal{P} = \{\mu_1, \dots, \mu_m\}. \quad (1.27)$$

Furthermore the weights  $\mu_i$  in  $\mathcal{P}$  are linearly independent and the set  $P$  of prime polynomials,  $\xi_{\mu_i}$ ,  $i = 1, \dots, m$ , are algebraically independent. In addition, one has a bijection

$$\Lambda(S(n)^N) \rightarrow (\mathbb{N})^m, \quad \nu \mapsto (d_1(\nu), \dots, d_m(\nu)) \quad (1.28)$$

such that, writing  $d_i = d_i(\nu)$ , up to scalar multiplication,

$$\xi_\nu = \xi_{\mu_1}^{d_1} \cdots \xi_{\mu_m}^{d_m} \quad (1.29)$$

and (1.29) is the prime factorization of  $\xi_\nu$  for any  $\nu \in \Lambda(S(\mathfrak{n})^N)$ . Finally,

$$S(\mathfrak{n})^N = \mathbb{C}[\xi_{\mu_1}, \dots, \xi_{\mu_m}] \quad (1.30)$$

so that  $S(\mathfrak{n})^N$  is a polynomial ring in  $m$ -generators.

**Remark 4.** One may readily extend part of Theorem 7 to weight vectors in  $Q(\mathfrak{n})^N$ . In fact one easily establishes that there is a bijection

$$\Lambda(Q(n_-)^N) \rightarrow (\mathbb{Z})^m, \quad \gamma \mapsto (e_1(\gamma), \dots, e_m(\gamma))$$

so that writing  $e_i(\gamma) = e_i$  one has

$$\xi_\gamma = \xi_{\mu_1}^{e_1} \cdots \xi_{\mu_m}^{e_m}. \quad (1.31)$$

Separating the  $e_i$  into positive and negative sets yields  $\xi_\nu$  and  $\xi_\mu$  of (1.24).

**1.6.** Let  $\nu \in \Lambda(S(\mathfrak{n})^N)$ . Then by Theorem 6 and (1.25) one has

$$\nu \in \Lambda_{\mathcal{B}} \cap \Lambda_{\text{dom}}$$

so that there exists nonnegative integers  $b_\beta$ ,  $\beta \in \mathcal{B}$  such that

$$\nu = \sum_{\beta \in \mathcal{B}} b_\beta \beta. \quad (1.31a)$$

**Remark 5.** The nonnegativity follows from dominance since one must have  $(\nu, \beta) \geq 0$  for  $\beta \in \mathcal{B}$ .

We wish to prove

**Theorem 8.** *One has*

$$\sum_{\beta \in \mathcal{B}} b_\beta = \deg \nu, \quad (1.32)$$

and as a function  $\xi_\nu \mid \mathfrak{r}_-^\times$  does not vanish identically and up to a scalar

$$\xi_\nu \mid \mathfrak{r}_-^\times = \prod_{\beta \in \mathcal{B}} e_\beta^{b_\beta}. \quad (1.33)$$

**Proof.** Let  $S^{\deg \nu}(\mathfrak{n})(\nu)$  be the  $\nu$  weight space in  $S^{\deg \nu}(\mathfrak{n})$ . It does not reduce to zero since  $\xi_\nu \in S^{\deg \nu}(\mathfrak{n})(\nu)$ . Let  $\Gamma$  be the set of all maps  $\gamma : \Delta_+ \rightarrow \mathbb{N}$  such that

$$\begin{aligned} \sum_{\varphi \in \Delta_+} \gamma(\varphi) &= \deg \nu \\ \sum_{\varphi \in \Delta_+} \gamma(\varphi) \varphi &= \nu. \end{aligned} \quad (1.34)$$

Then if

$$e^\gamma = \prod_{\varphi \in \Delta_+} e_\varphi^{\gamma(\varphi)},$$

the set  $\{e^\gamma \mid \gamma \in \Gamma\}$  is clearly a basis of  $S^{\deg \nu}(\mathfrak{n})(\nu)$  and consequently unique scalars  $s_\gamma$  exist so that

$$\xi_\nu = \sum_{\gamma \in \Gamma} s_\gamma e^\gamma. \quad (1.35)$$

But by Theorem 5 there exists  $x \in X$  such that  $\xi_\nu(x) \neq 0$ . However since  $X$  is  $B$ -homogeneous, the  $H$ -orbit  $\mathfrak{r}_-^\times$  is contained in  $X$  and there exists  $t \in \mathfrak{r}_-^\times$  such that  $x = u \cdot t$  for some  $u \in N$ . But since  $\xi_\nu$  is  $N$ -invariant one has  $\xi_\nu(t) \neq 0$ . But from (1.34) this implies that

$$\sum_{\gamma \in \Gamma} s_\gamma e^\gamma(t) \neq 0. \quad (1.36)$$

But  $e^\gamma(t) = 0$  for any  $\gamma \in \Gamma$  such that  $\gamma(\varphi) \neq 0$  for  $\varphi \notin \mathcal{B}$ . Thus there exists  $\gamma' \in \Gamma$  such that

$$\gamma'(\varphi) = 0$$

for all  $\varphi \notin \mathcal{B}$  and

$$e^{\gamma'}(t) \neq 0. \quad (1.37)$$

But by the independence of  $\mathcal{B}$  one has that  $\gamma'$  is unique and hence one must have  $\gamma'(\beta) = b_\beta$ . A similar argument yields (1.33). QED

## 2. A representation-theoretic construction, due to Lipsman–Wolf, of certain elements in $S(\mathfrak{n})^N$

**2.1.** Let  $\lambda \in \Lambda_{\text{dom}}$  and let  $V_\lambda$  be a finite-dimensional irreducible  $\mathfrak{g}$ -module with highest weight  $\lambda$ . Then, correspondingly,  $V_\lambda$  is a  $U(\mathfrak{g})$ -module with respect to a surjection  $\pi_\lambda : U(\mathfrak{g}) \rightarrow \text{End } V_\lambda$ . Let  $0 \neq v_\lambda \in V_\lambda$  be a highest weight vector. Also let  $V_\lambda^*$  be



the contragredient dual  $\mathfrak{g}$ -module. The pairing of  $V_\lambda$  and  $V_\lambda^*$  is denoted by  $\langle v, z \rangle$  with  $v \in V_\lambda$  and  $z \in V_\lambda^*$ . (We will use this pairing notation throughout in other contexts.) But as one knows  $V_\lambda^*$  is  $\mathfrak{g}$ -irreducible with highest weight  $\lambda^* \in \Lambda_{\text{dom}}$  given by

$$\lambda^* = -w_o \lambda. \quad (2.1)$$

But then by (1.1) and the mutual orthogonality of roots in the cascade

$$-\lambda^* = \lambda - \sum_{\beta \in \mathcal{B}} \lambda(\beta^\vee) \beta.$$

That is

$$\lambda + \lambda^* = \sum_{\beta \in \mathcal{B}} \lambda(\beta^\vee) \beta \quad (2.2)$$

and hence

$$\lambda + \lambda^* \in \Lambda_{\mathcal{B}} \cap \Lambda_{\text{dom}}. \quad (2.3)$$

On the other hand, regarding  $U(\mathfrak{g})^*$  as a  $\mathfrak{g}$ -module (dualizing the adjoint action on  $U(\mathfrak{g})$ ) it is clear that if  $f \in U(\mathfrak{g})^*$  defined by putting, for  $u \in U(\mathfrak{g})$ ,

$$f(u) = \langle u v_\lambda, z_{\lambda^*} \rangle, \quad (2.4)$$

then

$$\begin{aligned} f &\text{ is } \mathfrak{n}\text{-invariant and} \\ f &\text{ is an } \mathfrak{h} \text{ weight vector of weight } \lambda + \lambda^*. \end{aligned} \quad (2.5)$$

Now it is true (as will be seen below) that  $\lambda + \lambda^* \in \Lambda(S(\mathfrak{n})^N)$ . It is the idea of Lipsman–Wolf to construct  $\xi_{\lambda+\lambda^*}$  using  $f$ . The method in  $[L - W]$  is to symmetrize  $f$  and restrict to  $S(\mathfrak{n})$ . However Lemma 3.7 in  $[L - W]$  is incorrect (one readily finds counterexamples). But the idea is correct. One must modify  $f$  suitably and this we will do in the next section.

**2.2.** Assume  $\mathfrak{s}$  is a finite-dimensional Lie algebra. Let  $U_j(\mathfrak{s})$ ,  $j = 1, \dots$ , be the standard filtration of the enveloping algebra  $U(\mathfrak{s})$ . Let  $0 \neq f \in U(\mathfrak{s})^*$ . We will say that  $k \geq -1$  is the codegree of  $f$  if  $k$  is maximal such that  $f$  vanishes on  $U_{k-1}(\mathfrak{s})$ . But then if  $k$  is the codegree of  $f$  and if  $x_i \in \mathfrak{s}$ ,  $i = 1, \dots, k$ , and  $\sigma$  is any permutation of  $\{1, \dots, k\}$ , then  $(x_1 \cdots x_k - x_{\sigma(1)} \cdots x_{\sigma(k)}) \in U_{k-1}(\mathfrak{s})$  so that

$$f(x_1 \cdots x_k) = f(x_{\sigma(1)} \cdots x_{\sigma(k)}). \quad (2.6)$$

But this readily implies that there exists a unique element  $f_{(k)} \in S^k(\mathfrak{s})$  such that for any  $u \in U_k(\mathfrak{s})$  one has

$$f_{(k)}(\tilde{u}) = f(u) \quad (2.7)$$

where  $\tilde{u} \in S^k(\mathfrak{s})$  is the image of  $u$  under the Birkhoff–Witt surjection  $U_k(\mathfrak{s}) \rightarrow S^k(\mathfrak{s})$ .

Now let  $\mathfrak{s} = \mathfrak{g}$  and let  $f$  be given by (2.4). Let  $k$  be the codegree of  $f$ . Identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  using the Killing form. Then  $f_{(k)} \in (S^k(\mathfrak{g}))^N$  and is an  $H$ -weight vector of weight  $\lambda + \lambda^*$ . On the other hand, by (1.2),

$$U_k(\mathfrak{g}) = U_k(\mathfrak{n}_-) \oplus U_{k-1}(\mathfrak{g})\mathfrak{b}. \quad (2.8)$$

However  $\mathfrak{b} \cdot v_\lambda \subset \mathbb{C} v_\lambda$  so that  $f$  vanishes on  $U_{k-1}(\mathfrak{g})\mathfrak{b}$ . But this readily implies  $f_{(k)} \in S(\mathfrak{n})^N$ . We have proved

**Theorem 9.** *Let  $f$  be given by (2.4) and let  $k$  be the codegree of  $f$ . Then  $\lambda + \lambda^* \in \Lambda(S(\mathfrak{n})^N)$ . Furthermore  $k = \deg(\lambda + \lambda^*)$  and up to scalar multiplication*

$$f_{(k)} = \xi_{\lambda+\lambda^*}. \quad (2.9)$$

The inclusion (1.25) is actually an equality

$$\Lambda(S(\mathfrak{n})^N) = \Lambda_{\text{dom}} \cap \Lambda_{\mathcal{B}}. \quad (2.10)$$

This equality is due to Tony Joseph and I was not aware of it until read it in [J]. However, the equality (2.10) follows immediately from the modified Lipsman–Wolf construction Theorem 9. Indeed let  $\nu \in \Lambda_{\text{dom}} \cap \Lambda_{\mathcal{B}}$ . To show  $\nu \in \Lambda(S(\mathfrak{n})^N)$ , it suffices to show that

$$e_i(\nu) \geq 0 \quad (2.11)$$

in (1.31) for any  $i = 1, \dots, m$ . But putting  $\lambda = \nu$ , one has  $\lambda + \lambda^* = 2\nu$  and by Theorem 9 one has all  $e_i(2\nu) \geq 0$ . But clearly  $e_i(2\nu) = 2e_i(\nu)$ . This proves (2.11).

The results in this paper will appear in [K1] in Progress in Mathematics, in honor of Joe.

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